10 - Implication

Implication ("if p then q" or "$p \Rightarrow q$" or "p implies q") models valid deductions:

$$\text{"}(x=y) \overset{\times c}{\Longrightarrow} (cx = cy)\text{"} \text{ is true.}$$

So all instances of this is true:

$$(1=1) \overset{\times 2}{\Longrightarrow} (2=2) \quad \text{... "T => T" is true.}$$

$$(0=1) \overset{\times 2}{\Longrightarrow} (0=2) \quad \text{... "F => F" is true.}$$

$$(0=1) \overset{\times 0}{\Longrightarrow} (0=0) \quad \text{... "F => T" is true.}$$

But:

"If 1=1, then 0=1." is false: valid deductions never makes true things false.

Truth table for =>:

p	q	p => q
T	T	T
T	F	F
F	T	T
F	F	T

"If I'm truth-teller, I tell truths." is true.
"If I'm truth-teller, I tell lies." is false.
"If I'm a liar, I tell truths." is true.
"If I'm a liar, I tell lies." is true.

Think: Liars are unpredictable.

Example 1.

"If $\underbrace{\text{pigs could fly}}_{\text{hypothesis}}$, then $\underbrace{\text{I can swim}}_{\text{conclusion}}$." ... (F=>F) = True, so is (F=>T): with absurd (false) premise, we can deduce anything.

"If 6 divides 4686, then 3 divides 4686." ... True: either check "6 divides 4686" & "6 divides 4686" are true, so (T=>T) evaluates to True, or just note that this is a valid deduction: if 6 divides a number, so does its factor 3.

Example 2. Job promises that "If you show up to work Monday, then you get the job." But this turned out to be false. What can you conclude?

You showed up to work Monday but you don't get the job.
Example 3 formalizes the logic of negating implications of Example 2.

Example 3. Show $\neg(p \Rightarrow q) \equiv p \wedge \neg q$

p	q	$p \Rightarrow q$	$\neg(p \Rightarrow q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

p	q	$\neg q$	$p \wedge (\neg q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

Both $\neg(p \Rightarrow q)$ and $p \wedge (\neg q)$ show the same column of Trues and Falses (F,T,F,F), so these propositions are logically equivalent.

Example 4. Negate "If my car broke, then I cannot come."

My car is not broke and I cannot come.

More translations of $p \Rightarrow q$:
"p only if q", "q whenever p", "q is necessary for p", "p is sufficient for q"

Order of operations:

Highest			Lowest
$\neg$	$\wedge, \vee, \oplus$	$\Rightarrow$	$\Leftrightarrow$

and read from left to right

Example 5. Fully parenthesize the following propositions.

$p \vee q \Leftrightarrow p \equiv (p \vee q) \Leftrightarrow p$

$\neg p \Leftrightarrow p \Leftrightarrow \neg(p \Leftrightarrow p) \equiv ((\neg p) \Leftrightarrow p) \Leftrightarrow (\neg(p \Leftrightarrow p))$

$p \Rightarrow q \Rightarrow r \equiv (p \Rightarrow q) \Rightarrow r$

$p \Rightarrow q \Rightarrow r \wedge s \equiv (p \Rightarrow q) \Rightarrow (r \wedge s)$

Definition. The converse of p=>q is q=>p. The contrapositive is ~q=>~p.

Example 6. "If my car broke, then I cannot come."
Converse: "If I cannot come, then my car broke."
Contrapositive: "If I can come, then my car did not break."

Exercise 3b shows the contrapositive is equivalent to the original implication.

Optional Homework due March 25th or 26th.
Show your work. Answer without work receives no credit.

1. Write "Stop, or I'll shoot" as an implication.
2. Negate the if-then statement: "If Sara lives in Athens, then she lives in Greece."
3. Show (a) $p \Rightarrow q \equiv \neg p \vee q$, (b) $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$, (c) $(p \Rightarrow q) \Rightarrow q) \equiv p \vee q$

Non-Homework Problems.
4. (Very hard) Translate "Children and seniors pay half price" and "Children or seniors pay half price" into logically equivalent statements. (Hint: a straightforward English-to-logic translation will not work; instead try letting C="You are a child", S="You are a senior", P="You pay half price". Then somehow translate each of the two sentences with the help of implications.)